of arbitrary shape in an otherwise isothermal cavity. In more general terms, these effects will occur in the radiation from a blackbody surface at wavelengths which are of the order of the body dimensions.

Finally, it is worth while to recall that simply by invoking the thermodynamic principle of the detailed balancing of radiation one can compute the spectral absorptivity of a body from its spectral thermal radiation pattern or vice versa. Thus, it follows that the spectral absorptivity of a body also exhibits the above-itemized characteristics at wavelengths which are of the order of the body dimensions.

V. Acknowledgment

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Variational Principles and Mode Coupling in Periodic Structures*

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Summary—**Variational techniques are used in analyzing periodic "cold" microwave structures for the angular frequency, ω, as a function of assumed phase shift per periodic cell. Two variational expressions are given: one for ω in terms of the E- and H-fields, and one for $k^2=\omega^2\mu\epsilon$ in terms of the E-field. For structures with relatively light coupling between cells, the trial fields to be used with the variational expressions are composed of closed cavity modes, phase shifted by ϕ radians from cell to cell. Both variational expressions yield determinantal equations for $k^2(\phi)$ which agree with equations previously derived from a mode coupling point of view. One form of an equivalent lumped circuit is given to represent the structure within one of its pass bands.**

Curves compare the variational-mode coupling expression for $k^2(\phi)$ of a periodically lumped loaded transmission line with exact expressions.

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I. Introduction

Periodic structures have received extensive theoretical and experimental treatment in the past in regard to applications in traveling wave tubes. The advent of high power traveling wave tubes proved it was still not a simple matter to design slow wave structures which could dissipate large amounts of power while still operating over a relatively large band of frequencies [1].

To analyze a simple periodic structure, such as an iris-loaded waveguide, is a formidable task. The method of analysis used here is that of mode coupling. For the case of a very heavily shunt-loaded waveguide, the coupling holes in the irises are small compared to the total iris area. The structure resembles a chain of loosely-coupled resonant cavities and so the electric and magnetic fields of the structure resemble normal cavity modes of a section of the structure. The weaker the coupling between sections is made, the closer is the resemblance of the fields to those of the actual cavities.

In this mode-coupling analysis, the fields of the periodic structure are approximated by the normal cavity modes of oscillation of a section of the structure. These cavity modes are phase shifted from section to section by a phase angle ϕ to simulate a propagating wave. By use of these approximations, a relation between ω^2 (ω=frequency) and phase shift per section, ϕ, may be derived. In this way, the propagation characteristics of the heavily loaded structure are obtained in terms of resonant modes of a cavity section of the structure for any assumed phase shift ϕ. The coupling coefficients between adjacent sections are expressed in an intuitively appealing form in this mode coupling approach.

In solving for the physical parameters of a very complicated system, we often find it advantageous to use variational techniques to arrive systematically at good approximate answers. Such techniques have been used to find cutoff frequencies and propagation constants of waveguides as well as resonant frequencies for cavities. In all of these applications, variational principles allowing the use of trial fields which are not exactly correct in detail can give quite accurate results.

The periodically loaded waveguide, viewed either as a waveguide with slight periodic perturbations of its walls or as a chain of coupled cavities, can be treated using a variational principle. The variational principle used here allows ω^2 to be computed for assumed trial fields and phase shift per section ϕ. For the case of heavy shunt loading of the guide, trial fields which are actually resonant modes of a cavity section of the structure are used together with an assumed ϕ. The relation between ω^2 and ϕ that is found is identical to the coupling equations derived purely from a mode coupling approach [2], [3].

The variational technique is appealing because it yields a sort of optimum value of ω^2 for the given set of trial functions, namely that ω^2 which is minimized for these trial functions. But this implies that the error in ω^2 is of second order compared to the actual error in the trial functions, and this is the advantage of this technique. The trial fields are not just used as approximate fields to get answers; they may be adjusted according to a definite procedure to obtain the most accurate approximation to the correct ω^2 for the assumed form of the trial fields.

The fact that the same coupling equations are found from mode coupling and from the variational principle signifies that the mode-coupling formalism has a formal mathematical basis in variational techniques. Mode coupling, then, is not simply an intuitively convenient approximation to use in some problems.

The results obtained here for the heavily-loaded periodic structure are useful in studying the effects of more complicated schemes of coupling between adjacent cavity sections. The mode-coupling equations for a capacitively lumped-loaded transmission line were derived, and for this simple periodic structure it is easy to obtain curves to show how well very simple trial fields may be used to obtain accurate answers.

II. The Variational Principle

In a lossless periodic structure such as the iris-loaded waveguide shown in Fig. 1, we know that the electric and magnetic fields satisfy Maxwell's equations

$$\nabla \times \boldsymbol{E} + j\omega\mu \boldsymbol{H} = 0, \tag{1}$$

$$\nabla \times \boldsymbol{H} - j\omega\epsilon \boldsymbol{E} = 0, \tag{2}$$

when the medium filling the structure is isotropic, homogeneous, and nonconducting (a time-dependence of $e^{j\omega t}$ has been assumed). The boundary condition that the electric field satisfies is stated as

$$\boldsymbol{n} \times \boldsymbol{E} = 0 \qquad \text{on } S, \tag{3}$$

where $\boldsymbol{n}$ is the unit vector which is normal to the perfectly conducting surface S which makes up the periodic structure, (see Fig. 2). Another important property of the fields in a periodic structure is expressed in a theorem by Floquet [4]. From this theorem, we may write in functional notation

$$\boldsymbol{E}(u_1, u_2, z) = \widehat{\boldsymbol{E}}(u_1, u_2, z)e^{-\gamma z}, \tag{4}$$

where now $\widehat{\boldsymbol{E}}(u_1, u_2, z)$ is a periodic function of z having the same period L as the structure. (u_1 and u_2 are generalized transverse coordinates.)

$$\widehat{\boldsymbol{E}}(u_1, u_2, z) = \widehat{\boldsymbol{E}}(u_1, u_2, z + L). \tag{5}$$

The magnetic field naturally has this same property.

Using these known properties of the fields of the periodic structure, one may derive a variational expression for $j\omega$. Consider the $\boldsymbol{E}$- and $\boldsymbol{H}$-fields of (1) and (2) to pertain to a wave traveling in the positive z-direction

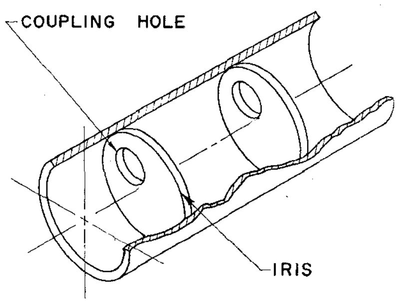

Fig. 1—An iris-loaded waveguide.

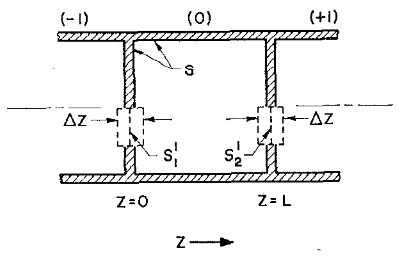

Fig. 2—A section of the periodic structure showing notation used.

of the structure. Calling these fields E_+ and H_+, we may now dot-multiply (1) by H_-^* (*stands for complex conjugate), the magnetic field of another wave solution called the adjoint solution, and integrate throughout the volume V of one cell or section of the structure. Correspondingly, multiply (2) by E_-^*, the electric field of the adjoint solution, and integrate. Subtracting the two resulting equations and solving for $j\omega$, which is not a function of the variables of integration, gives

$$j\omega = \frac{\int_V E_-^* \cdot \nabla \times H_+ dV - \int_V H_-^* \cdot \nabla \times E_+ dV}{\mu \int_V H_+ \cdot H_-^* dV + \epsilon \int_V E_+ \cdot E_-^* dV}. \quad (6)$$

The proof that (6) is actually a variational principle for ω is given in Appendix I. It is found in this proof that for arbitrary first-order variation of ω equal to zero, the adjoint solution satisfies the equations

$$\nabla \times E_-^* - j\omega\mu H_-^* = 0 \quad (7)$$

$$\nabla \times H_-^* + j\omega\epsilon E_-^* = 0, \quad (8)$$

provided the adjoint trial field satisfies boundary condition (3) and the conditions $E_-(\text{L}) = E_-(0)e^{-\gamma L}$, $H(\text{L}) = H-(0)e^{-\gamma L}$. Since the latter conditions imply Floquet's theorem, we can demand

$$E_-(u_1, u_2, z) = E_-(u_1, u_2, z)e^{-\gamma z}, \quad (9)$$

where this is the same value of γ as in (4), which indicates that the adjoint wave also propagates in the positive z-direction.

The variational principle may be used together with suitable trial fields for E_+, H_+, E_-, and H_- to obtain approximations to ω which are in error to second order compared to the errors in the trial fields themselves. Consider first an iris-loaded waveguide in which the coupling holes in the irises are very small. The coupling between adjacent cells is now very weak and the periodic structure actually appears to be a chain of weakly-coupled cavities. The fields in a particular cavity section of the periodic structure should resemble to a fair degree the fields of an undriven cavity. For this case then, the pass bands are very narrow compared to the stop bands; and since we are interested in the propagation characteristics in a pass band only, we are concerned with a narrow frequency range. It seems reasonable, then, to approximate each field by several terms of a normal mode expansion of a cavity section. This is what we will do to construct trial fields. The simplest approximation would be to use only one term of a mode expansion as a trial field.

However there are two different normal mode expansions which can be defined for a cavity section. The short-circuit mode expansion is derived for the boundary condition that the tangential electric field disappears over the entire surface of the cavity including the surfaces S_1', S_2' (see Fig. 2) in the plane of the holes. The open-circuit mode expansion is found for the boundary condition of (3) on S (but the tangential magnetic field must be zero on S_1', S_2'). These two mode expansions are closely related to the behavior of the periodic structure. For every short-circuit resonant mode, there is a corresponding open-circuit mode which has fields very similar to the short-circuit mode except in the immediate vicinity of the coupling holes. The resonant frequencies of this pair of modes are not very different when the holes are small. The frequency range between them corresponds to a pass band of the periodic structure [4]. To obtain an approximation to the propagation characteristics in a particular pass band, we would choose the short-circuit mode corresponding to that pass band as the E_+ and H_+ trial fields and the open-circuit mode as the E_- and H_- trial fields.

Let us summarize the properties of these two mode expansions. First, the fields of the nth short-circuit mode can conveniently be defined by the equations [5]:

$$\nabla \times E_n = P_n H_n, \quad (10a)$$

$$\nabla \times H_n = P_n E_n, \quad (10b)$$

$$P_n^2 = \omega_{sn}^2 \mu\epsilon, \quad (11)$$

$$n_0 \times E_n = 0 \qquad \text{on } S, S_1', S_2', \quad (12)$$

where n_0 is the outward unit vector normal to the boundary surface of a cavity section. ω_{sn} is the resonant frequency of this nth short-circuit mode. For the mth open circuit mode the resonant frequency is ω_{om} and the corresponding equations are:

$$\nabla \times e_m = p_m h_m, \quad (13a)$$

$$\nabla \times h_m = p_m e_m, \quad (13b)$$

$$p_m^2 = \omega_{om}^2 \mu\epsilon, \quad (14)$$

$$n_0 \times e_m = 0 \qquad \text{on } S, \quad (15a)$$

$$n_0 \times h_m = 0 \qquad \text{on } S_1', S_2'. \quad (15b)$$

We have seen from Appendix I that the trial fields must also satisfy Floquet's theorem, so the E_+ trial field is constructed as follows. Remember that we are trying to approximate the fields of the periodic structure for frequencies in the nth pass band for a propagating wave having a phase shift per section of ϕ radians. The electric field in the (0) cavity section of Fig. 2 is approximated by the cavity short-circuit mode E_n. The field in the preceding (-1) cavity section is assumed to be the same as in the (0) section, but phase shifted by an angle ϕ to simulate propagation in the positive z-direction,

$$E_+^{(0)} = V_n E_n; \qquad E_+^{(-1)} = V_n E_n e^{j\phi}. \quad (16a)$$

The H_+ trial field is similarly constructed from H_n.

$$H_+^{(0)} = I_n H_n; \qquad H_+^{(-1)} = I_n H_n e^{j\phi}. \quad (16b)$$

V_n and I_n are merely adjustable amplitude factors in the trial fields which will be adjusted in an optimum way using the variational principle.

For the adjoint trial fields, propagation in the positive z-direction is also simulated, and we use the open-circuit mode corresponding to this nth pass band in which we are interested.

$$E_-^{(0)} = v_n e_n; \qquad E_-^{(-1)} = v_n e_n e^{j\phi} \tag{17a}$$

$$H_-^{(0)} = i_n h_n; \qquad H_-^{(-1)} = i_n h_n e^{j\phi}. \tag{17b}$$

These trial fields are now substituted into (6) and a relation for $j\omega$ is found in terms of ϕ, the phase shift per section. An important point to notice is the fact that the trial fields are discontinuous in the plane of the holes because each cavity mode is actually a standing oscillation. The discontinuity is due to a phase shift and possibly to the behavior of the mode patterns (see Figs. 3 and 4). If the surfaces S_1' and S_2' are considered as pillbox volumes only Δz thick in which the discontinuities appear, we must choose the volume, V, of one section so that it includes part of the pillbox at S_1' and part of the pillbox at S_2' so the total volume integral throughout a section contains the volume of one whole pillbox (see Fig. 2).

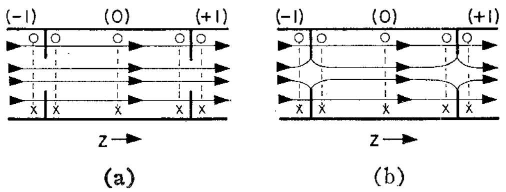

Fig. 3—Examples of (a) an even short-circuit mode and (b) an even open-circuit mode. Solid lines represent electric field lines; dashed lines represent magnetic field lines.

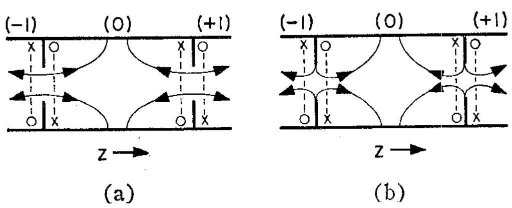

Fig. 4—Examples of (a) an odd short-circuit mode and (b) an odd open-circuit mode.

In (6), the integral

$$\int_V E_-^* \cdot \nabla \times H_+ dV$$

may be broken up into the integral over the volume of a section plus an integral over a pillbox. The pillbox volume is important in this term since there is a differentiation of field discontinuities in $\nabla \times H_+$. Since Δz is considered to be very small and the trial field to be substituted for E_-^* is tangential at S_1' and S_2', we may replace $\nabla \times H_+$ by $a_z \times \partial H_+ / \partial z$, which is the only term producing components tangential to S_1' and S_2'. The pillbox integral may be written as

$$\int_{S_1'} \left[\int_{Z=0^-}^{Z=0^+} e_n^* \cdot \nabla \times H_n dz \right] da$$

$$= \int_{S_1'} \left[\int_{Z=0^-}^{Z=0^+} e_n^* \cdot {}_z a \times \frac{\partial}{\partial z} H_n dz \right] da$$

$$= \frac{1}{2} \int_{S_1'} [e_n^*(0^-) + e_n^*(0^+)] \cdot [a_z \times H_n(0^+) - a_z \times H_n(0^-)] da. \tag{18}$$

The final result in (18) is obtained by our assuming any smooth functional variation of the fields e_n^* and H_n within the pillbox, so long as each of these fields has the same functional variation. The final result, then, does not depend on the specific functional variation, and the form in (18) emerges.

The cavity modes of a section of the periodic structure are either symmetric or antisymmetric with respect to the z-axis in a section. Let us define an "even" mode as one which has an even number of reversals of its electric field along the z-axis of a section. From Fig. 3, we can see that for even modes,

$$H_n{}^0(0^-) = H_n{}^{(-1)}(L) = H_n{}^{(-1)}(0) = H_n{}^{(0)}(0) e^{j\phi}, \tag{19a}$$

$$e_n{}^0(0^-) = e_n{}^{(-1)}(L) = - e_n{}^{(-1)}(0) = - e_n{}^{(0)}(0) e^{j\phi}. \tag{19b}$$

Let us solve for $\omega(\phi)$ between P_2 and p_2 by employing the E_2—H_2 and e_2—h_2 modes in (6). With the aid of (16), (17), and (18), the expression for ω is found to be

$$j\omega = \frac{v_2^* I_2 P_2 T_2 - v_2^* I_2 M_2(1 - \cos\phi) - i_2^* V_2 P_2 U_2}{\mu I_2 i_2^* U_2 + \epsilon V_2 v_2^* T_2} \cdot \tag{20}$$

We have defined

$$T_2 = \int_V e_2^* \cdot E_2 dV$$

$$U_2 = \int_V h_2^* \cdot H_2 dV$$

$$M_2 = \int_{S_1'} a_z \times e_2^*(0^+) \cdot H_2(0^+) da,$$

where a_z is the unit vector in the z-direction. To optimize the coefficients for the best approximation to ω we must minimize (20) with respect to each amplitude coefficient. Setting the partial derivatives of (20) with respect to i_2^* and v_2^* equal to zero gives the coupling equations

$$P_2 V_2 = - j\omega\mu I_2, \tag{21}$$

$$P_2 I_2 T_2 = j\omega\epsilon V_2 T_2 + I_2 M_2 (1 - \cos\phi). \tag{22}$$

Eliminating V_2 from these two expressions results in one equation

$$(P_2^2 - k^2) T_2 = P_2 M_2 (1 - \cos\phi), \tag{23}$$

where $k^2 = \omega^2 \mu \epsilon$. Eq. (23) also results from our varying (20) with respect to I_2 and V_2.

When the phase shift per section is π radians, we are at a cutoff of the pass band and also at a resonance of a cavity section. For even modes, $\phi = \pi$ radians corresponds to the open circuit resonance where then $k^2 = p^2$. Use of this information in (23) leads to[1]

$$\frac{P_2 M_2}{T_2} = \frac{1}{2}(P_2^2 - p_2^2). \tag{24}$$

Then (23) may be simplified to,

$$k^2 = \frac{p_2^2 + P_2^2}{2} - \frac{p_2^2 - P_2^2}{2} \cos \phi. \tag{25}$$

We define "odd" modes as those normal modes of a cavity section in which the electric field reverses an odd number of times along the z-axis. From Fig. 4, we can see that for odd modes (19a) and (19b) take the new form

$$\boldsymbol{H}_n^0(0^-) = \boldsymbol{H}_n^{(-1)}(L) = -\boldsymbol{H}_n^{(-1)}(0) = -\boldsymbol{H}_n^{(0)}(0)e^{j\phi}, \tag{26a}$$

$$\boldsymbol{e}_n^0(0^-) = \boldsymbol{e}_n^{(-1)}(L) = \boldsymbol{e}_n^{(-1)}(0) = \boldsymbol{e}_n^{(0)}(0)e^{j\phi}. \tag{26b}$$

Use of these expressions would give as a final result

$$k^2 = \frac{p_1^2 + P_1^2}{2} + \frac{p_1^2 - P_1^2}{2} \cos \phi, \tag{27}$$

where we use the odd subscript 1 to show that (27) applies for an odd mode pair.

If more cavity modes are included in the trial fields, one obtains a matrix equation whose determinant set equal to zero replaces the (23) for k^2. This determinant checks that previously derived from a mode-coupling approach [2]. It will also agree with the determinant later to be derived from a variational expression for k^2 in terms of the $\boldsymbol{E}$-field alone.

An equivalent circuit interpretation of (20) follows for both even and odd modes. Combining (19a) and (19b) with (26a) and (26b) results in a compact way to express the mode properties:

$$\boldsymbol{H}_n(L) = (-1)^n \boldsymbol{H}_n(0) = \boldsymbol{H}_n(0^-)e^{-j\phi} \tag{28a}$$

$$\boldsymbol{e}_m(L) = (-1)^{m+1}\boldsymbol{e}_m(0) = \boldsymbol{e}_m(0^-)e^{-j\phi}, \tag{28b}$$

where the subscripts n will be even to refer to even modes and odd when odd modes are used. If we use (28a) and (28b) in (6) and obtain a more general form of (20) valid for even or odd mode pairs we may vary this expression with respect to V_n, I_n, v_n^*, and i_n^* in a manner analogous to our operations with (20). We obtain a set of equations which leads to an equivalent circuit representation of the two-mode-coupling approximation (even or odd modes) for a periodic cavity structure. Now define

$$V_n = -j\omega L_n I_n$$

$$L_n = \frac{\mu}{P_n}; \qquad C_n = \frac{\epsilon}{P_n}. \tag{29}$$

We have now

$$V_n + \frac{I_n}{j\omega C_n \dfrac{T_n}{M_n}} - \frac{I_n}{j\omega C_n} - \frac{I_n e^{j\phi}}{2j\omega C_n \dfrac{T_n}{M_n}}(-1)^n - \frac{I_n e^{-j\phi}}{2j\omega C_n \dfrac{T_n}{M_n}}(-1)^n = 0. \tag{30}$$

Fig. 5 shows a lumped circuit which leads to the same (30) and is then one form of an equivalent circuit for the periodic structure. This equivalent circuit stems directly from the mode-coupling approach. For even modes, the quantity T_n/M_n is negative. For odd modes, T_n/M_n is positive and there is a negative capacitive coupling between cavity sections.

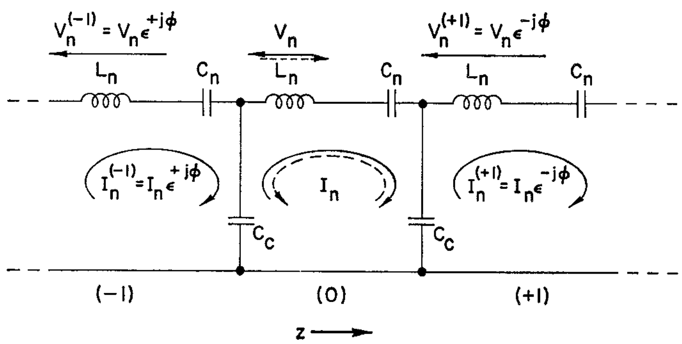

Fig. 5—A lumped equivalent circuit for the periodic structure from the single mode pair analysis. The dotted lines correspond to odd modes, solid lines to even modes. ($C_c = -2C_n T_n/M_n$).

In the variational principle derived here, both electric and magnetic fields appear, and this means both trial fields must be chosen, but according to certain constraints. There is another variational principle for periodic structures in which only a single field appears. The electric field of a periodic structure obeys the vector wave equation

$$\nabla \times \nabla \times \boldsymbol{E} - k^2 \boldsymbol{E} = 0. \tag{31}$$

We may consider the $\boldsymbol{E}$-field of (31) to pertain to a wave traveling in the positive z-direction and call it the $\boldsymbol{E}_+$ field. Then dot-multiply (31) by $\boldsymbol{E}_-^*$, the adjoint wave electric field, and integrate throughout the volume of one section of the structure. Solving for k^2, we get

[1] It is to be understood that this relation (24) is valid for a cavity section if the fields used for $\boldsymbol{E}_-^*$ and $\boldsymbol{E}_+$ in the integrals M_2 and T_2 are valid representations of the exact fields at its open- and short-circuit resonances, respectively. In this case of a lossless structure, only a single mode is used to represent the fields at each of these resonances, and (24) is exact. If complete open- and short-circuit normal mode expansions for the cavity section were used in M_2 and T_2, in general, (24) would be exactly true.

$$k^2 = \frac{\int_V E_-^* \cdot \nabla \times \nabla \times E_+ dV}{\int_V E_-^* \cdot E_+ dV}. \quad (32)$$

The actual proof that (32) is a variational principle for k^2 involving only electric fields is shown in Appendix II. It is found in this proof that setting $\delta[k^2]=0$ forces the adjoint solution to satisfy the equation

$$\nabla \times \nabla \times E_-^* - k^2 E_-^* = 0, \quad (33)$$

provided $E_-(L) = E_-(0)e^{-\gamma L}$. Since the latter condition implies (9), we see that the adjoint is a wave propagating in the positive z-direction. Now we need only to construct trial fields for E_+ and E_- for (32).

This time, let us construct more general trial fields for more accuracy in finding k^2. If the modes of either expansion were arranged in order of increasing resonant frequencies, one would find even and odd modes alternating. We could attach even or odd subscripts to correspond to even and odd modes, and then the mode properties could be written as in (28a) and (28b). The simple single mode-pair approximation using only a single term for each trial field gives good results when the coupling between cavities is weak. As the coupling holes are made larger, the actual fields do not resemble very closely a single resonant mode of a section. The next step in trial fields would be to make them more accurate by including several more terms of each expansion for trial fields. The extra terms should, of course, be chosen as those having resonant frequencies closest to the pass band of interest. The trial fields in the (0) cavity may be expressed as

$$E_+^{(0)} = a_1 E_1 + a_2 E_2 + \cdots + a_n E_n + \cdots = \sum_n a_n E_n, \quad (34a)$$

the lowest-order mode being even, as depicted in Fig. 3.

$$E_-^{(0)} = b_1 e_1 + b_2 e_2 + \cdots + b_m e_m + \cdots = \sum_m b_m e_m. \quad (34b)$$

Putting these trial fields [(34a) and (34b)] into (32) would give a coupling equation containing the coefficients a_n and b_m. To optimize these coefficients for the values of k^2 corresponding to the perturbed P_n and P_m, we would take partial derivatives with respect to each a_n (or b_m; the final determinantal equation is the same) and set the equations equal to zero. For the uth (a_u) coefficient, we would get

$$\sum_m \left[(P_u^2 - k^2)T_{um} - P_u M_{um} \left\{ \frac{1}{2} + \frac{1}{2}(-1)^{m+u} - \frac{e^{j\phi}}{2}(-1)^u - \frac{e^{-j\phi}}{2}(-1)^m \right\} \right] b_m^* = 0. \quad (35)$$

The following definitions have been introduced:

$$T_{um} = \int_V E_u \cdot e_m^* dV$$

$$M_{um} = \int_{S_1'} a_z \times e_m^*(0^+) \cdot H_u(0^+) da.$$

We could cast this set of equations into matrix form, which would yield the determinantal equation for k^2 in terms of ϕ.

$$\begin{bmatrix} [(P_1^2 - k^2)T_{11} - P_1 M_{11}(1+\cos\phi)] & [-P_1 M_{12}(j \sin\phi)] & \cdots \\ [-P_2 M_{21}(-j\sin\phi)] & [(P_2^2 - k^2)T_{22} - P_2 M_{22}(1-\cos\phi)] & \cdots \\ \vdots & \vdots & \end{bmatrix} \begin{bmatrix} b_1^* \\ b_2^* \\ \vdots \end{bmatrix} = 0 \quad (36)$$

The determinantal equation arising from (36) agrees exactly with that found by Bevensee [2], solving the problem without the use of the variational principle but merely using a straightforward mode-coupling approach. It can also be seen that (36) simplifies to (25) (with the "2" subscript changed to "0") or (27), if only M_{11} or M_{22} is assumed to be nonzero. This reduces the trial fields to the single mode-pair case again.

If we used the complete normal mode expansions in (34a) and (34b), we could represent the fields in a cavity section exactly at any frequency, and these trial fields, together with (32), would yield an exact determinantal equation for k^2. In this paper, we are studying the problems of approximating this determinantal equation for k^2 by using approximate but simple trial fields in (32).

III. The Periodically Lumped-Loaded Transmission Line

For illustration of the accuracy inherent in this mode-coupling approach to periodic structures, numerical results were worked out for a simple model of a periodic structure. The model chosen was a transmission line shunted periodically by lumped capacitance. Since the exact propagation characteristics for this model can easily be found, a comparison could be made between the exact behavior and that predicted by the mode-coupling equations [6], [7].

The periodically-loaded lossless transmission-line model that was found very convenient to handle using variational techniques was that of a nonuniform line in which the inductance and capacitance were allowed to vary (periodically) with distance along the line. Analogous to the Maxwell equations used for the periodic cavity structure, we now have the nonuniform transmission-line equations

$$\frac{d}{dz} V(z) = -j\omega L(z) I(z) \tag{37}$$

$$\frac{d}{dz} I(z) = -j\omega C(z) V(z), \tag{38}$$

where the time dependence $e^{j\omega t}$ is assumed again, and z is the distance along the line. It can be verified in a manner completely analogous to the above presentation that a variational principle for this periodic structure is

$$\omega^2 = -\frac{\int_0^L V_-^* \frac{d}{dz}\left[\frac{1}{L(z)} \frac{d}{dz} V_+\right] dz}{\int_0^L C(z) V_+ V_-^* dz}, \tag{39}$$

where L is the length of one periodic section of the structure.

Up to this point these expressions are perfectly general for a lossless, periodic, transmission-line structure. For simulation of periodic shunt loading by lumped capacitor elements, $L(z)$ is treated as a constant, L_0, and $C(z)$ takes the form of a constant, C_0, plus impulse functions (area $2C_s$) which appear periodically with separation L. This completes the one-dimensional formulation of this structure. Fig. 6 shows

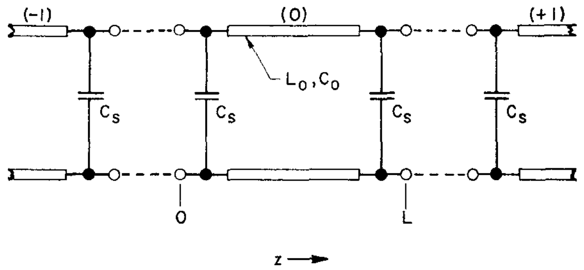

Fig. 6—The periodically-loaded transmission line broken into sections.

the structure that is analyzed when it is broken into convenient periodic sections. The open-circuit and short-circuit modes of a section of this structure are easily found, and these modes are used to approximate voltage and current distributions on the section. The open-circuit modes are substituted in (39) for V_- and the short-circuit modes for V_+.

Curves for the structure shown in Fig. 6 are shown only for two pass bands occurring at the lowest frequencies. When the trial fields consist of only single terms of the mode expansions discussed above, the curves are marked "single mode-pair case"; for trial fields of two terms from each expansion, the terminology "two mode-pair case" is used, etc.

The loading factor is defined as the ratio of the susceptance of one of the shunt capacitors to the characteristic admittance of the unloaded line at a normalized frequency of 1. Fig. 7 shows the ω vs ϕ characteristics for the first two pass bands for the various trial fields. Fig. 8 shows the ω vs ϕ curves for various loading factors (various amounts of shunt loading) for the first two pass bands.

Notice that in Fig. 7(b) the two mode-pair approximations give poorer results than does the single mode-pair case. When two modes are used for a trial field, one mode would be the one corresponding to the second pass band and the other would correspond to either the first or third pass band. But the corrections in ω resulting from the resonant modes corresponding to the first and third pass bands are of equal magnitude. Inclusion of the first and not the third pass band contribution causes more of an error than if both extra terms were not used.

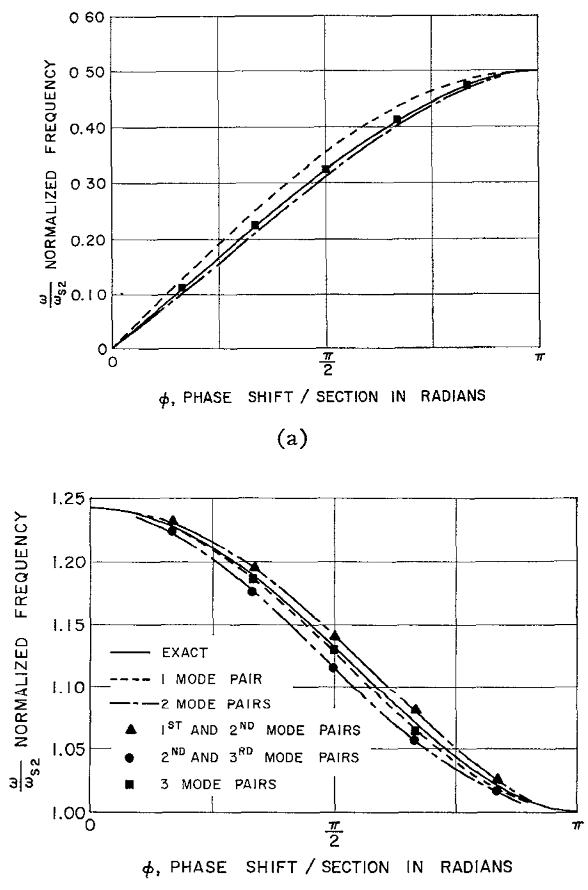

Fig. 7—(a) The ω vs ϕ characteristics for the periodically loaded transmission line for the first pass band. (b) The ω vs ϕ characteristics for the periodically loaded transmission line for the second pass band.

Notice that the three mode-pair case using resonant modes corresponding to all three pass bands is more accurate.

Fig. 8 indicates that as the shunt loading is increased, the mode-coupling equations give better results. This is due to the fact that coupling between adjacent sections of the structure is weaker and each section behaves more like an independent cavity. The trial fields then resemble the actual fields more closely and hence we get better results.

IV. Conclusions

It has been shown that a mode-coupling approach to the solution of a heavily-loaded periodic structure leads to simple and appealing equations. Moreover, the mode-coupling approach is shown to have as its formal mathematical basis the variational principle which allows the use of approximations in field configurations. Indeed, the use of a variational principle guarantees that the approximations resulting from the chosen trial field configurations are made in a systematic way so as to give the optimum value of ω or k^2 for a given phase shift per section, ϕ. It is legitimate to make the small errors in choosing trial fields as long as one knows that these errors lead to errors of lower order in the final results, the values of ω or k^2.

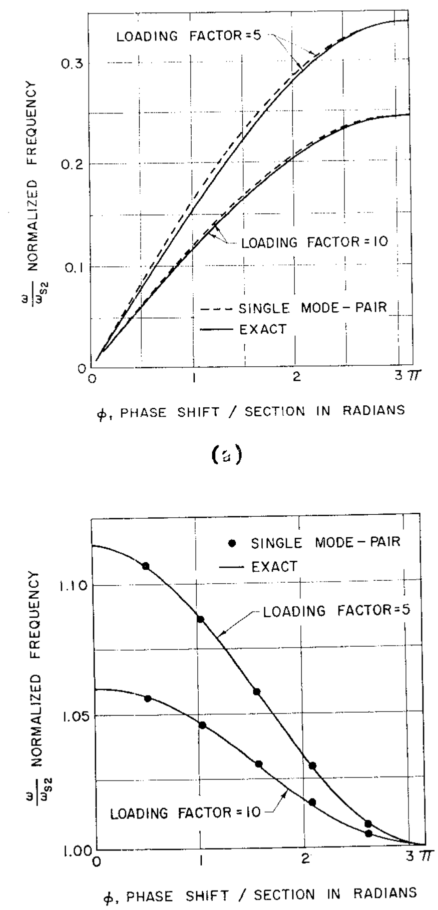

Fig. 8—(a) The ω vs ϕ characteristics for two loading factors for the first pass band. (b) The ω vs ϕ characteristics for two loading factors for the second pass band.

Also, the mode-coupling or variational technique did lead to an acceptable and simple equivalent lumped circuit for a periodic structure with heavy shunt loading. This form of equivalent circuit has been assumed many times without the mathematical verification for its validity.

The single mode-pair coupling coefficient between sections of the periodic structure has the form

$$\frac{\int_{S_1'} \boldsymbol{a}_z \times \boldsymbol{e}_m^*(0^+) \cdot \boldsymbol{H}_n(0^+) da}{\int_V \boldsymbol{e}_m^* \cdot \boldsymbol{E}_n dV},$$

which gives a clear indication as to the mechanism involved in this coupling. What one must do to increase or decrease or change the sign of this coupling is also clear. Very complicated coupling schemes have been studied from the point of view of mode coupling and it is easy to predict qualitative behavior of these complicated structures [6].

One must realize that mode coupling as used here, is an intuitive concept, in that it allows the intuition to help in getting an approximate solution to a complicated problem. Mode coupling provides the trial fields which are then used with a variational principle to arrive at mathematically-sound expressions for approximate answers.

Appendix I

It is to be understood that (6) is a variational principle for ω if it can be shown that for first-order variations of the trial fields from the exact fields, the first-order variation in ω vanishes. There remain only second- and higher-order variations in ω. Consider $\boldsymbol{E}_+$ to be the exact field existing in the periodic structure. We do not know the details of this exact field solution, but we can make reasonable guesses from experience as to what the field should look like. As a trial field, we then do not use $\boldsymbol{E}_+$ because we don't know it, but we use a trial field which may be expressed as the exact field plus a small functional variation,

$$\boldsymbol{E}_+^T = \boldsymbol{E}_+ + \delta \boldsymbol{E}_+. \tag{40}$$

We wish to see how the variations of all the fields appearing in (6) affect the value of ω. Notice that the variational notation δ used as an operator is commutative with respect to differentiation, so we can perform the following type of transformation,

$$\int_V E_-^* \cdot \nabla \times \delta H_+ dV = \int_V \delta H_+ \cdot \nabla \times E_-^* dV + \oint n_0 \times \delta H_+ \cdot E_-^* da. \quad (41)$$

We can then find the first-order variation of $\delta[\omega]$ to be

$$j\delta[\omega]\left[\mu \int_V H_+ \cdot H_-^* dV + \epsilon \int_V E_+ \cdot E_-^* dV\right]$$
$$= \int_V \delta E_-^* \cdot (\nabla \times H_+ - j\omega\epsilon E_+) dV$$
$$- \int_V \delta H_-^* \cdot (\nabla \times E_+ + j\omega\mu H_+) dV$$
$$- \int_V \delta E_+ \cdot (\nabla \times H_-^* + j\omega\epsilon E_-^*) dV$$
$$+ \int_V \delta H_+ \cdot (\nabla \times E_-^* - j\omega\mu H_-^*) dV$$
$$- \oint n_0 \times E_-^* \cdot \delta H_+ da - \oint n_0 \times \delta E_+ \cdot H_-^* da. \quad (42)$$

Since we know that the E_+ and H_+ (exact) fields satisfy Maxwell's equations, the first two terms on the right-hand side of (42) vanish. Also, we see that the adjoint solution is the wave solution that must satisfy

$$\nabla \times E_-^* - j\omega\mu H_-^* = 0 \quad (43)$$

$$\nabla \times H_-^* + j\omega\epsilon E_-^* = 0, \quad (44)$$

if the third and fourth terms on the right-hand side of (42) are to vanish. We are further constrained to have as a boundary condition on the adjoint solution

$$n_0 \times E_-^* = 0 \qquad \text{on } S, \quad (45)$$

and we must also be careful in choosing trial fields so that

$$n_0 \times \delta E_+ = 0 \qquad \text{on } S. \quad (46)$$

In order to show that $\delta\omega$, the first-order variation of ω, vanishes for arbitrary variations of the fields, we must deal with the surface integrals over the plane of the coupling holes S_1' and S_2'.

One way to make these surface integrals vanish is to have the integrals over S_1' cancel those over S_2'. If we choose trial fields which satisfy Floquet's theorem, then we see that if the adjoint solution satisfies Floquet's theorem we have for one of the integrals,

$$\int_{S_1'} n_0 \times E_-^*(0) \cdot \delta H_+(0) da + \int_{S_2'} n_0 \times E_-^*(L) \cdot \delta H_+(L) da. \quad (47)$$

But n_0 on S_2' is the unit vector a_z and n_0 on S_1' is $-a_z$. Also, we choose trial fields so that

$$\delta H_+(L) = \delta H_+(0) e^{-\gamma L}. \quad (48)$$

Then, if we have

$$E_-(L) = E_-(0) e^{-\gamma L}, \quad (49$$

it is seen the surface integral S_1' cancels the one over S_2' for γ imaginary. Also, we are forced to have

$$\delta E_+(L) = \delta E_+(0) e^{-\gamma L}$$
$$H_-(L) = H_-(0) e^{-\gamma L}, \quad (50)$$

in order that the last integral in (42) be zero. This means that the adjoint fields satisfy Floquet's theorem and so satisfy all the constraints that the original fields E_+ and H_+ do. But it is important to notice that the trial fields must be chosen so that the variations in the E_+ and H_+ fields satisfy Floquet's theorem and this will be fulfilled if the trial fields themselves satisfy Floquet's theorem. Under all these constraints, (6) is a variational principle for ω.

Notice the important point in this mathematical development that although the adjoint solution satisfies the same constraints that the original fields do, we are not constrained to make the adjoint the same as the original solution. This degree of freedom in choosing the adjoint solution is the key to the treatment of the periodic cavity structure.

Appendix II

To show that (32) is actually a variational principle for k^2, we find the first-order variation in k^2 using the usual variational operational notation. Notice that the double transformation of the integral below leads to two surface integrals.

$$\int_V E_-^* \cdot \nabla \times \nabla \times \delta E_+ dV = \int_V \nabla \times \delta E_+ \cdot \nabla \times E_-^* dV$$
$$+ \oint n_0 \times (\nabla \times \delta E_+) \cdot E_-^* da$$
$$= \int_V \delta E_+ \cdot \nabla \times \nabla \times E_-^* dV$$
$$+ \oint n_0 \times \delta E_- \cdot (\nabla \times E_-^*) da$$
$$+ \oint n_0 \times (\nabla \times \delta E_+) \cdot E_-^* da. \quad (51)$$

By use of (51) a variation of (32), yields

$$\left[\int_V E_+ \cdot E_-^* dV\right] \delta[k^2]$$
$$= \int_V \delta E_-^* \cdot [\nabla \times \nabla \times E_+ - k^2 E_+] dV$$

$$+\int_V \delta \boldsymbol{E}_+ \cdot [\nabla \times \nabla \times \boldsymbol{E}_+^* - k^2 \boldsymbol{E}_-^*] dV$$

$$+\oint \boldsymbol{n}_0 \times (\nabla \times \delta \boldsymbol{E}_+) \cdot \boldsymbol{E}_-^* da$$

$$+\oint \boldsymbol{n}_0 \times \delta \boldsymbol{E}_+ \cdot (\nabla \times \boldsymbol{E}_-^*) da. \qquad (52)$$

If we substitute into (52) the exact solution for the periodic structure, $\delta \boldsymbol{E}_+ = 0$ everywhere and we see that

$$\delta[k^2] = 0. \qquad (53)$$

Then, (32) is a variational principle for k^2 but it remains for us to find the constraints on the $\boldsymbol{E}_-$ field. Clearly, if $\delta \boldsymbol{E}_+$ is not zero, we must have

$$\nabla \times \nabla \times \boldsymbol{E}_-^* - k^2 \boldsymbol{E}_-^* = 0 \qquad (54)$$

to make the right-hand side of (52) zero. A boundary condition that

$$\boldsymbol{n}_0 \times \boldsymbol{E}_-^* = 0 \qquad \text{on } S \qquad (55)$$

is also required, as well as the constraint of (5). We are constrained to choose the trial field so that

$$\boldsymbol{n}_0 \times \delta \boldsymbol{E}_+ = 0 \qquad \text{on } S, \qquad (56)$$

in addition to the properties for $\boldsymbol{E}_-$ found in Appendix I. Also, if the trial field $\boldsymbol{E}_+^T$ is constructed so as to satisfy Floquet's theorem, we see that the surface integrals vanish again and (32) is a variational principle for k^2.

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